A discrete approach to monogenic analysis through Radon transform

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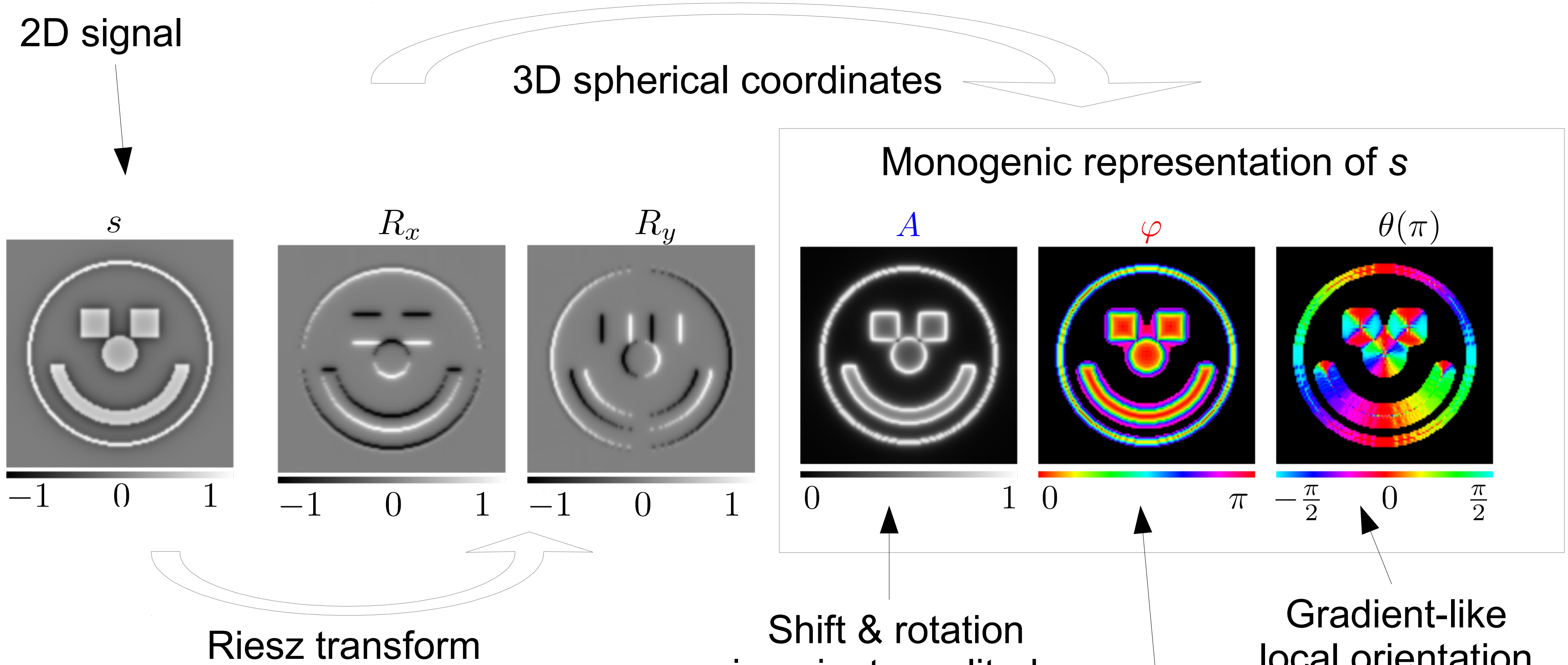
Outline

- Monogenic representation
- Radon transform and discrete geometry
- Radon domain monogenic analysis
- Discussion

Monogenic representation



$$[R_x\ R_y] = \frac{1}{2\pi}\, p.v. \int s(\boldsymbol{x}-\boldsymbol{\tau}) \frac{[\tau_1\ \tau_2]}{|\boldsymbol{\tau}|^3}\, d\tau$$

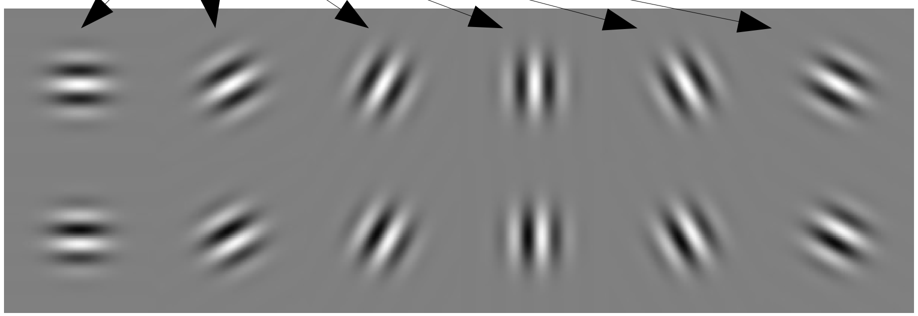
- Generalization of the *analytic signal* (Hilbert transform): “Oriented 1D phase analysis”
- Unified contour detector
- Signal should be narrowband: Wavelets ?

Monogenic representation

Gabor wavelets :
- complex coefficients: amplitude/phase
- “K” orientations

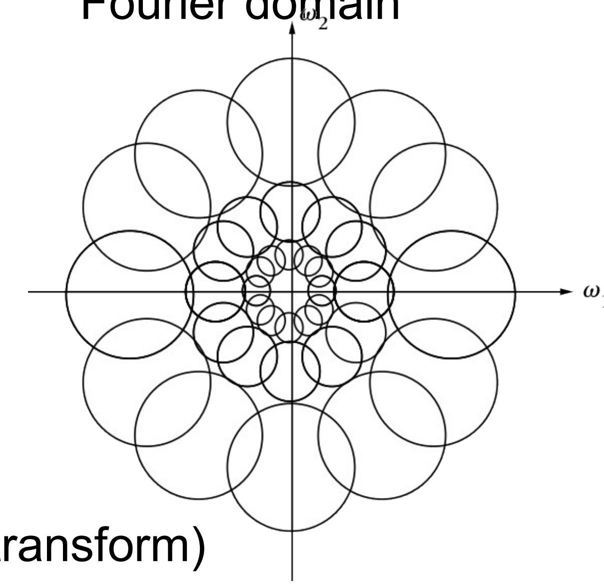
$\mathbb{C}$

Fourier domain

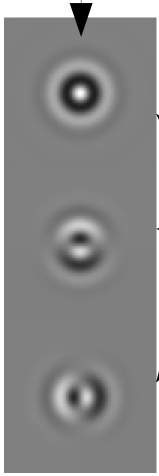


Real part

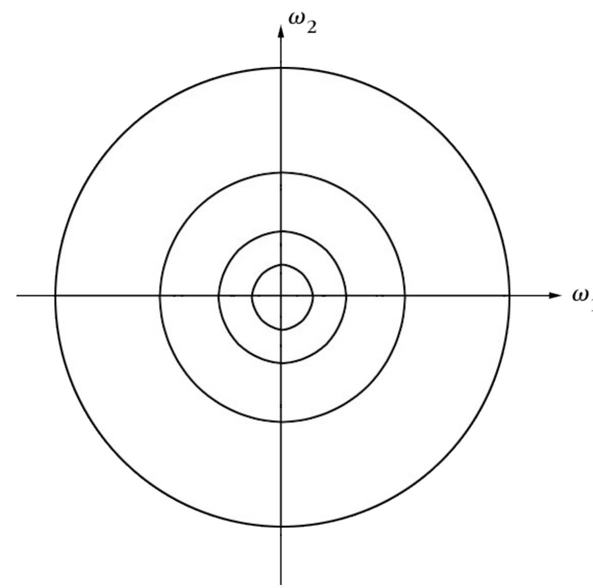
Imaginary part
(partial Hilbert transform)



Monogenic wavelets :
- 3-vector coefficients -> amplitude/phase/orientation
- 1 orientation, **isotropic** wavelet

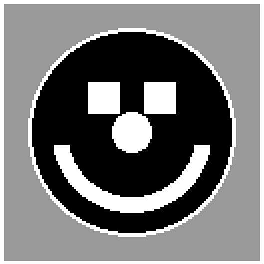


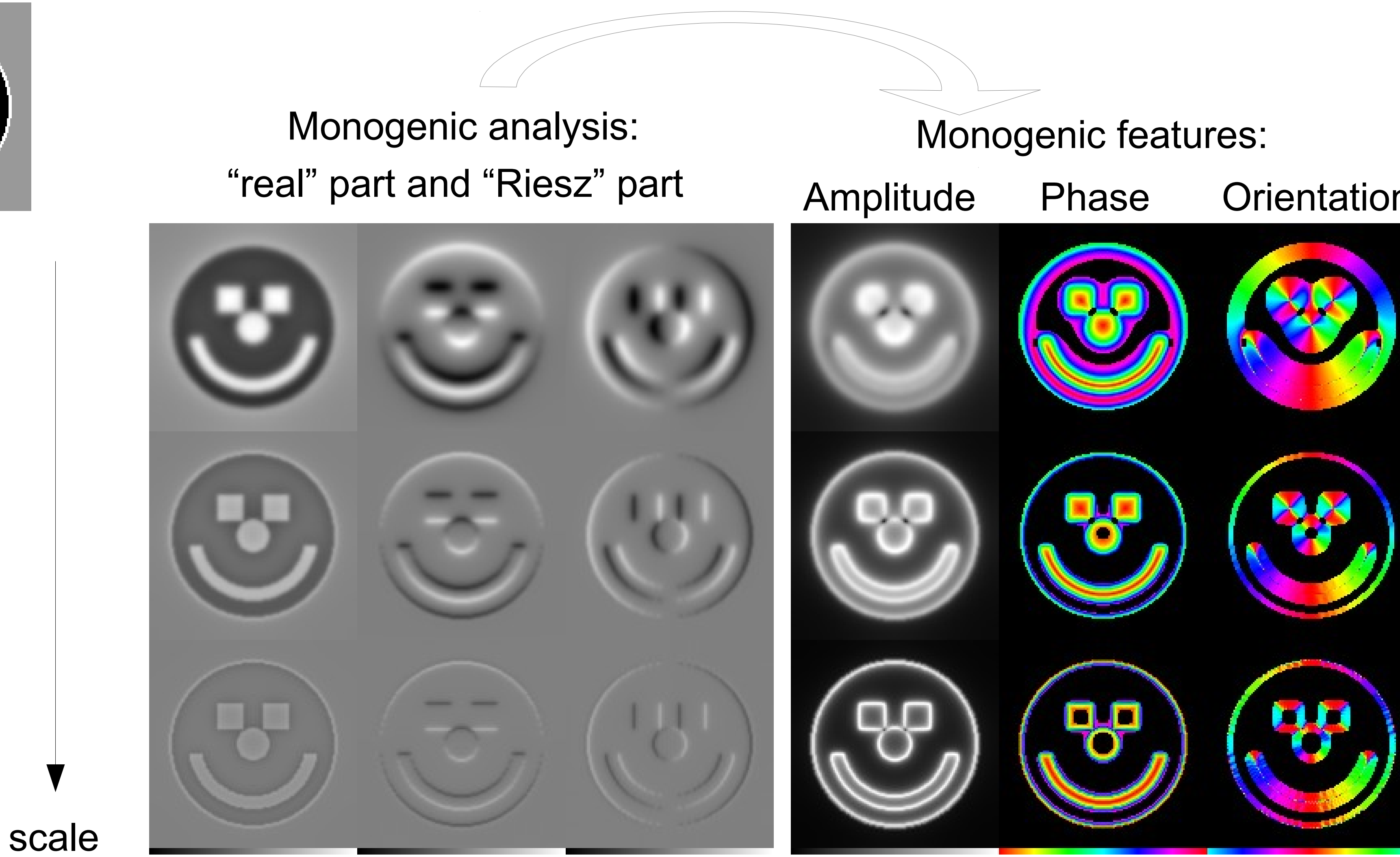
3 parts
(isotropic part and
its Riesz transform)



Monogenic representation

2D signal





Discrete world

- How to keep “continuous” properties?
- Isotropy / Cartesian grid?
- Perfect reconstruction from wavelet coefficients?
- Redundancy?

Riesz transform: infinite & continuous impulse response

In practice: **sampling of the impulse response**
→ discrete phase?
→ discrete orientation / isotropy?
→ invariance of amplitude?

Need for a discrete scheme with controlled approximation
of the continuous monogenic framework

Discrete world

1) Continuous link between 3 tools:
Riesz transf. ↔ **Hilbert** transf. in the **Radon** domain

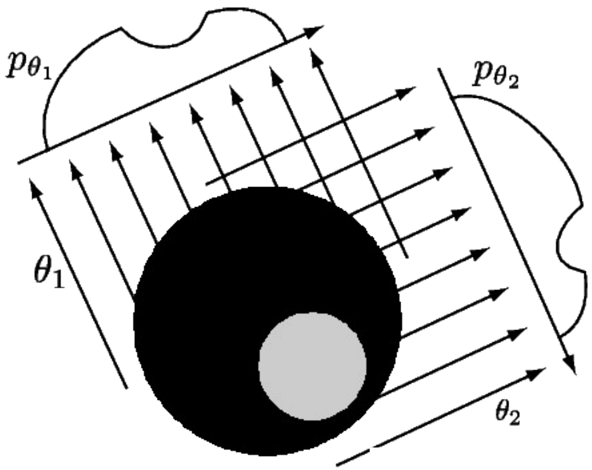
1D tool (simpler than
2D Riesz transform)

2) Well established **discrete** “Dual-Tree” complex wavelets
→ approximate Hilbert transform
→ approximate invariance of amplitude
→ perfect reconstruction
→ little redundancy

3) Well established **discrete** Radon transform
→ controlled approximate isotropy
→ perfect reconstruction
→ controlled redundancy

Radon transform

Oriented 1D projections of the signal:



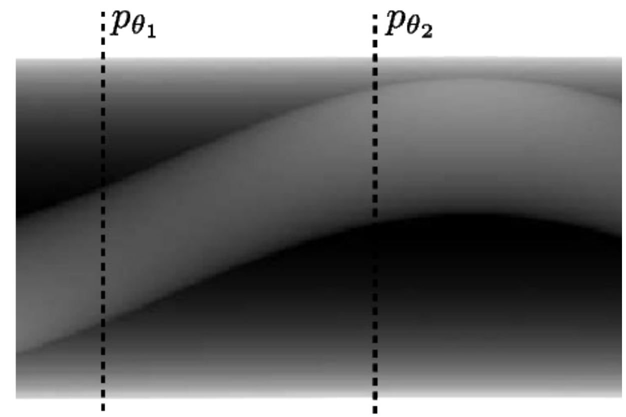


Illustration from S. Mallat, *A Wavelet Tour of Signal Processing*, 3rd Ed.

$$s_\theta(t) = \int_{\mathbb{R}} s(\ \tau \sin\theta + t\cos\theta\ ,\ -\tau\cos\theta + t\sin\theta\ ) d\tau$$

1D FT ↓

1: Fourier slice theorem: $\hat{s_\theta}(f) = \hat{s}(f\cos\theta, f\sin\theta)$ ← "Slices" of 2D FT of *s*

→ Direct link between 1D Hilbert analysis and 2D monogenic analysis

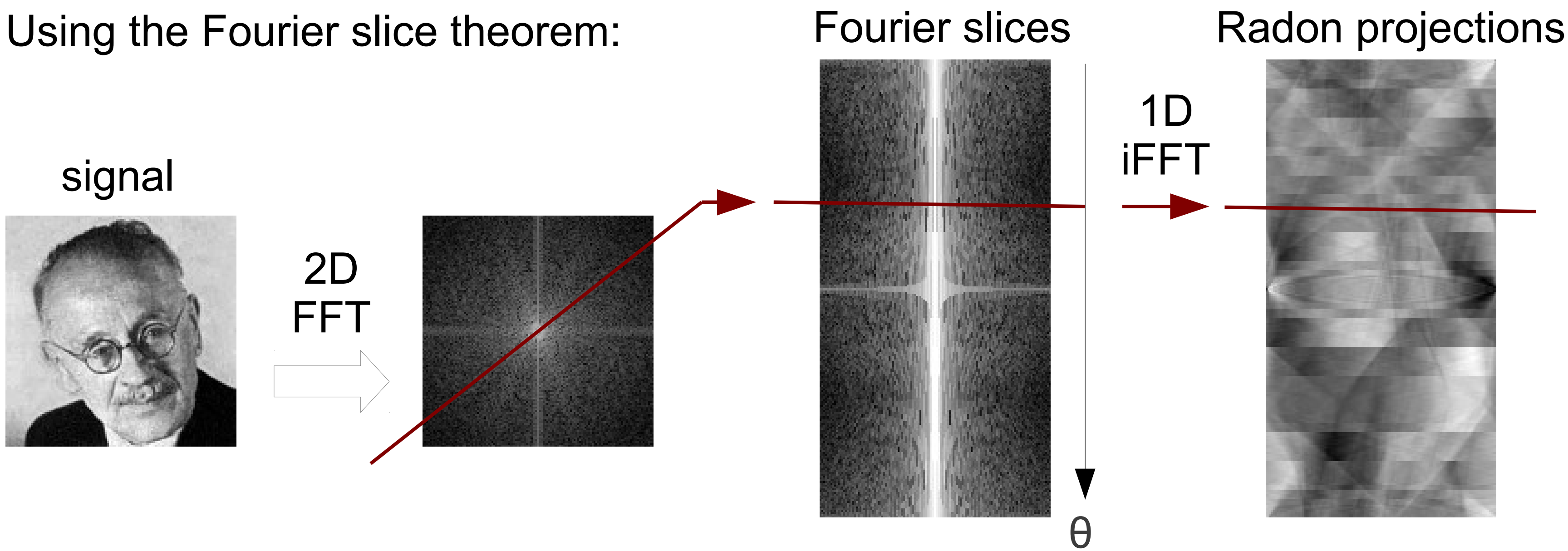
2: Existing **discrete** Radon transforms

3: Existing **discrete** 1D Hilbert analysis (complex wavelets)

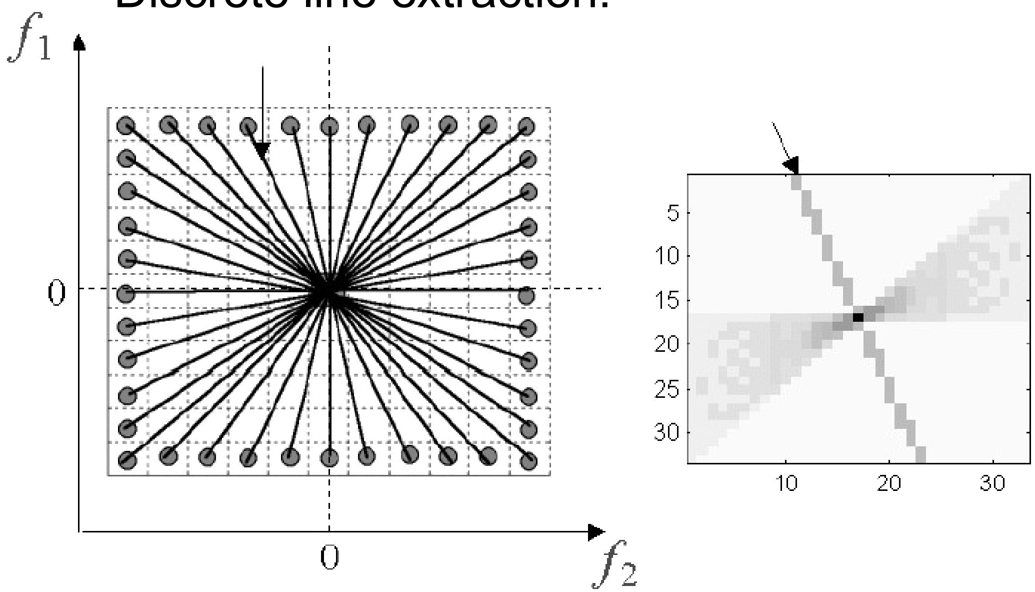
→ Propose a Radon-based discrete monogenic analysis

Radon transform based on discrete geometry

Using the Fourier slice theorem:



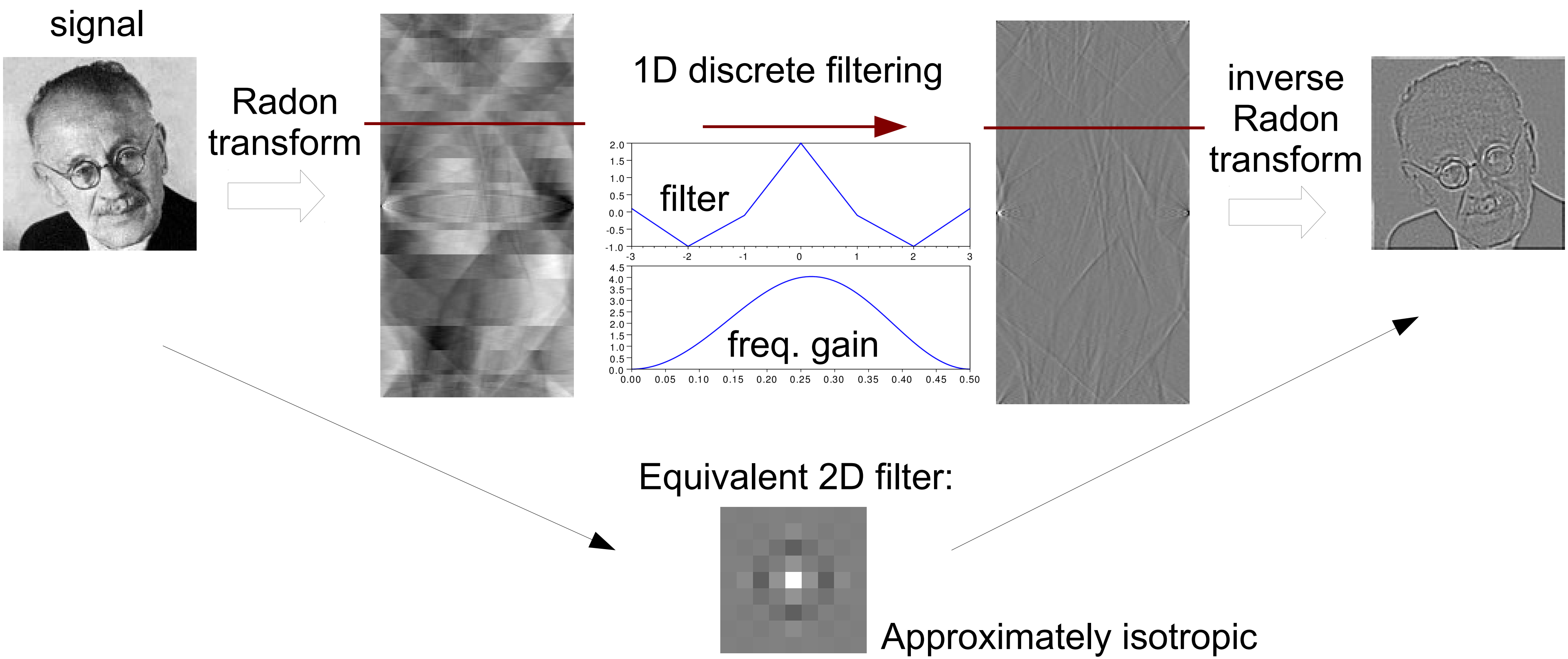
Discrete line extraction:



- Controlled overlapping
- Thickness parameter / Redundancy
- Exact inverse transform
- Straightforward 3D extension

[P. Carré & E. Andres, Discrete Analytical Ridgelet Transform, Sig. Proc., Elsevier, 2004]

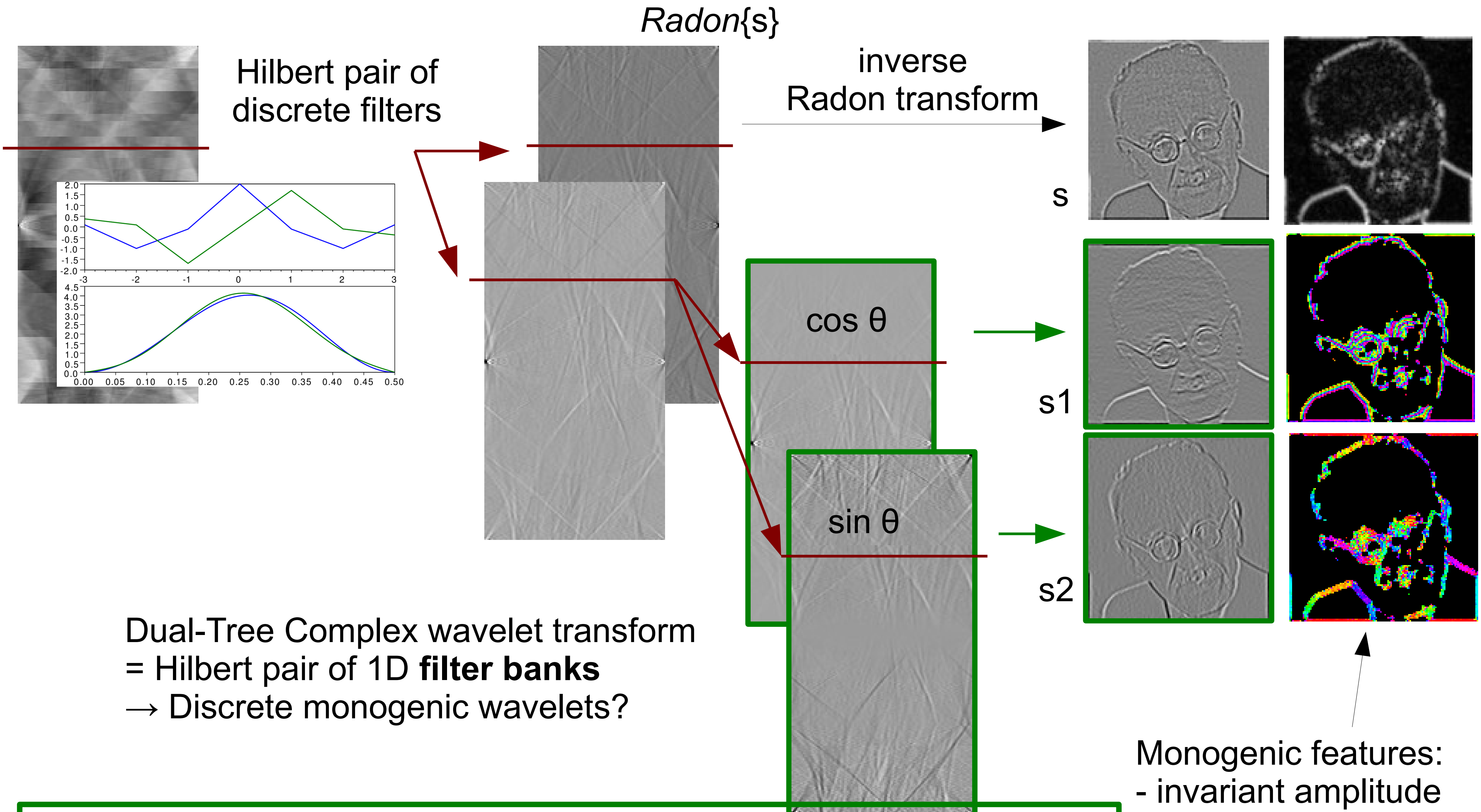
Radon domain filtering



Radon domain <u>Riesz transform</u> :

$$\textit{Riesz}\{ s \} = [\ s1 \ \ s2\ ]^T = \textit{Radon}^{-1}\Big\{ \textit{Hilbert}\{ \textit{Radon}\{s\} \} [\ \cos\theta \quad \sin\theta\ ]^T \Big\}$$

Radon domain Riesz transform



Dual-Tree Complex wavelet transform
= Hilbert pair of 1D **filter banks**
→ Discrete monogenic wavelets?

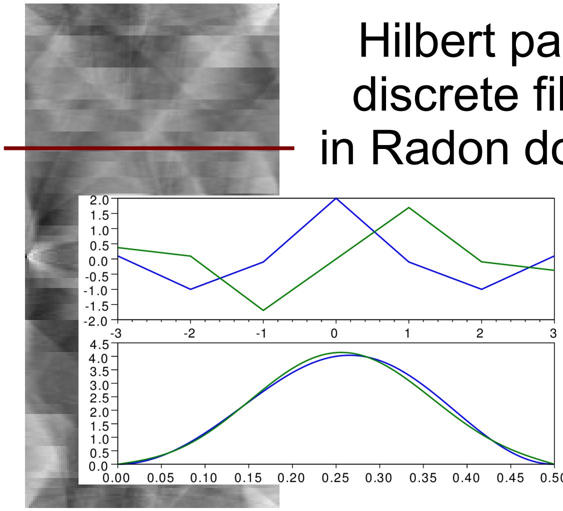
Monogenic features:
- invariant amplitude
- Contour classification
- Orientation analysis

Radon domain Riesz transform :

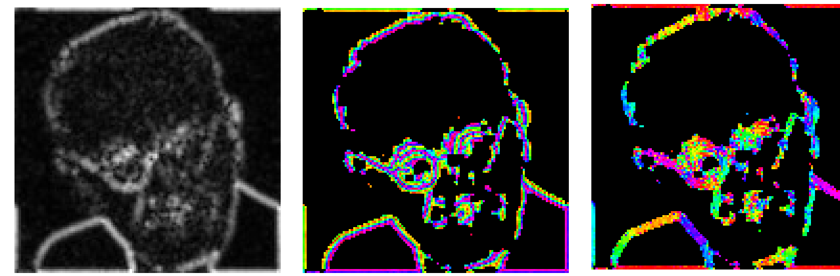
$$\textit{Riesz}\{ s \} = [\, s1 \;\; s2 \,]^T = \textit{Radon}^{-1}\Big\{ \textit{Hilbert}\{ \textit{Radon}\{s\} \} \, [\, \cos\theta \quad \sin\theta \,]^T \Big\}$$

Exact reconstruction

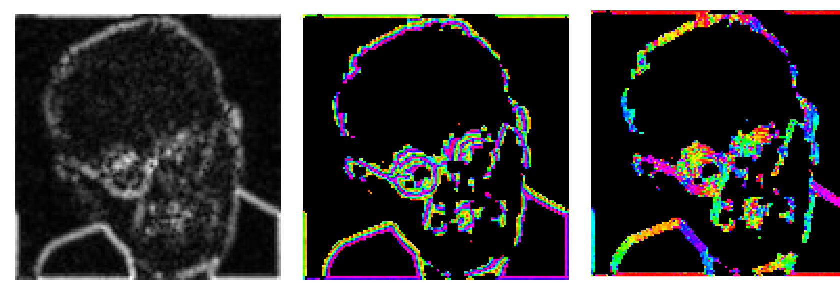
Discussion



Hilbert pair of
discrete filters
in Radon domain



Sampling of the
Riesz impulse response



- Obtained data is similar
- Discrete wavelet design may be simplified (because 1D)
- The scheme is fully reversible
- up- and down- sampling in the Radon domain must be studied

Conclusion

- Innovative discrete approach for monogenic analysis of images
- Use of a true discrete scheme of Radon transform
- Fully reversible discrete monogenic analysis for 1 scale
- Potential definition of efficient monogenic wavelets through existing 1D Hilbert pairs of filterbanks
- Need for a definition of Radon domain down-sampling (Dyadic/Quincunx?)

Thanks for your attention